

SPECHT PROPERTY FOR THE 2-GRADED IDENTITIES OF THE JORDAN SUPERALGEBRA OF A BILINEAR FORM

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ABSTRACT. Let K be a field of characteristic zero and V a vector space of dimension $m > 1$ with a nondegenerate symmetric bilinear form $f : V \times V \rightarrow K$. The Jordan algebra $B_m = K \oplus V$ of the form f is a superalgebra with this decomposition. We prove that the ideal of all the 2-graded identities of B_m satisfies the Specht property and we compute the 2-graded cocharacter sequence of B_m .

1. INTRODUCTION

Let K be a field of characteristic zero and V be a vector space of dimension $m > 1$ with a nondegenerate symmetric bilinear form f . The Jordan algebra of the form f (see [20]) is denoted by B_m . The algebra B_m is important in the class of Jordan algebras due to a well-known result of Zelmanov [19] that classified, up to isomorphism, finite dimensional simple Jordan algebras when K is algebraically closed in one of the following types: $M_n(K)^+$, the special Jordan algebra of $n \times n$ matrices, $M_n(K)^t$, the algebra of $n \times n$ symmetric matrices with respect to the transpose involution, $M_{2n}(K)^s$, the $2n \times 2n$ symmetric matrices with respect to the symplectic involution, and the algebra B_m when the form is nondegenerate. These are special simple Jordan algebras, for example, the Clifford algebra of the form f is an associative enveloping for B_m . The Jordan algebra B of the form f is defined similarly by replacing a vector space of finite dimension by a vector space of infinite dimension.

Polynomial identities for these algebras have been studied by a number of researchers. Drensky in [4] obtained a complete description of the relatively free algebras of the varieties generated by B_m and B denoted by $\text{var}(B_m)$ and $\text{var}(B)$ respectively. Furthermore he also established the GL_m -module structure of these algebras and their Hilbert series. Koshlukov has studied the polynomial identities and the asymptotic behavior of codimensions for the subvarieties of $\text{var}(B_m)$ and $\text{var}(B)$ (see [10]). The ideal $\text{Id}(A)$ of all identities of an algebra A satisfies the Specht property if $\text{Id}(A)$ itself and all T -ideals containing $\text{Id}(A)$ are finitely generated as T -ideals. Kemer proved that every associative algebra satisfies the Specht property (see, for example, [9]). Il'tyakov [8] proved that the variety of unitary algebras generated by B_m satisfy the Specht property. The analogous result for finitely generated Jordan PI-algebras was proved by Vais and Zelmanov in [15]. In [16] Vasilovsky has explicitated a finite bases for polynomial identities of B and B_m for $m \geq 2$ over an infinite field of characteristic $\neq 2$ using ideas developed by Il'tyakov in [8].

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Other types of identities such as trace identities and graded identities for the algebra B_m were also studied (see [17], [18], [11]). The decomposition $B_m = K \oplus V$ is a natural \mathbb{Z}_2 -grading and a finite basis for the graded identities was determined in [11]. The \mathbb{Z}_2 -graded identities of the algebras B_n with other types of gradings were studied in [17]. We remark that the group gradings of the algebras B_n were described in [2]. This result was generalized by Martino in [12] for $J_n = B_m \oplus D_k$ the Jordan algebra with a degenerate bilinear form of rank m , where D_k is the vector space spanned by the degenerate elements of the basis of a vector space of dimension n . For associative algebras graded by a finite group it was proved that every ideal of graded identities is finitely generated ([14], [1]). Concerning Lie algebras the Specht property for the graded identities of the Lie algebra $sl_2(K)$ of 2×2 traceless matrices was established in [13] for the trivial grading and in [6] for the other gradings.

In the present paper we prove that the variety of \mathbb{Z}_2 -graded Jordan algebras generated by $B_m = K \oplus V$ satisfies the Specht property and we compute the 2-graded cocharacter sequence of B_m .

2. PRELIMINARIES

Throughout the paper K is a field of characteristic zero and all vector spaces and algebras are considered over K . Let V_m be a vector space of dimension m with a nondegenerate symmetric bilinear form $f : V \times V \rightarrow K$. The vector space $B_m = K \oplus V_m$ with the multiplication defined by

$$(\alpha + u)(\beta + v) = (\alpha\beta + f(u, v)) + (\alpha v + \beta u), \quad \alpha, \beta \in K, u, v \in V,$$

is a Jordan algebra. We set $(B_m)_0 = K$, $(B_m)_1 = V_m$ and obtain the structure of a Jordan superalgebra on B_m .

Let $Z = X \cup Y$ be the disjoint union of two countable sets. The free Jordan algebra $J(Z)$ freely generated by Z has a \mathbb{Z}_2 -grading

$$J(Z) = J(Z)_0 \oplus J(Z)_1,$$

where $J(Z)_0$ (resp. $J(Z)_1$) is the subspace generated by the monomials with an even (resp. odd) number of variables in Y .

A homomorphism $\Theta : A = A_0 \oplus A_1 \rightarrow B = B_0 \oplus B_1$ of \mathbb{Z}_2 -graded algebras is *graded* if $\Theta(A_i) \subseteq B_i$, $i \in \mathbb{Z}_2$. A mapping $\theta : Z \rightarrow A$ such that $\theta(X) \subseteq A_0$ and $\theta(Y) \subseteq A_1$ can be uniquely extended to a graded homomorphism $\Theta : J(Z) \rightarrow A$. The set of polynomials $f(x_1, \dots, x_k, y_1, \dots, y_n)$ such that $f(a_1, \dots, a_k, a'_1, \dots, a'_n) = 0$ for every $a_1, \dots, a_k \in A_0$, $a'_1, \dots, a'_n \in A_1$ is an ideal of $J(Z)$. Such ideal, denoted by $\text{Id}_2(A)$ is invariant under all graded endomorphisms of $J(Z)$ and is called a *T_2 -ideal*. An element of $\text{Id}_2(A)$ is a *graded polynomial identity* for A . If $S \subseteq \text{Id}_2(A)$ is a set of polynomials such that $\text{Id}_2(A)$ is the intersection of all T_2 -ideals of $J(Z)$ containing S we say that this set is a *basis* for $\text{Id}_2(A)$. The variety of \mathbb{Z}_2 -graded Jordan algebras determined by A satisfies the *Specht property* if every T_2 -ideal containing $\text{Id}_2(A)$ admits a finite basis.

The T_2 -ideal $\text{Id}_2(A)$ is graded, i.e.,

$$\text{Id}_2(A) = \text{Id}_2(A) \cap J(Z)_0 \oplus \text{Id}_2(A) \cap J(Z)_1,$$

and therefore the quotient algebra $J(Z)/\text{Id}_2(A)$, denoted by $J(A)$, admits a natural grading that makes the quotient map a graded homomorphism. The algebra $J(A)$ is the *relatively free algebra* in the variety determined by A .

The study of the graded identities of an algebra A in characteristic zero can be reduced to the study of the multilinear ones. For every $n + k \geq 1$ and $k \geq 0$ the vector space $P_{k,n}$ of the multilinear polynomials of degree $n + k$ in the variables $x_1, \dots, x_k, y_1, \dots, y_n$ is a $S_k \times S_n$ -module under the action

$$(\lambda, \mu)f(x_1, \dots, x_k, y_1, \dots, y_n) = f(x_{\lambda(1)}, \dots, x_{\lambda(k)}, y_{\mu(1)}, \dots, y_{\mu(n)})$$

and this in turn induces a structure of the $S_k \times S_n$ -module to the space

$$P_{k,n}(A) = \frac{P_{k,n}}{(P_{k,n} \cap \text{Id}_2(A))}.$$

Let

$$\chi_{k,n}(A) = \chi_{k,n}(P_{k,n}(A)) = \sum m_{\sigma,\tau} \chi_{\sigma} \otimes \chi_{\tau}$$

be the decomposition of $\chi_{k,n}(A)$ the (k, n) -th cocaracter of A into irreducible $S_k \times S_n$ -characters.

We recall some results about finite basis property (or well-quasi-ordering). For more detailed background see [7].

A relation $a \leq b$ on a set A is a quasi-order if it is reflexive and transitive. If B is a subset of a quasi-ordered set A , the closure of B , denoted by \overline{B} , is defined as $\overline{B} = \{a \in A : \exists b \in B \text{ such that } b \leq a\}$. We say that B is a closed subset when $B = \overline{B}$. The quasi-ordered set (A, \leq) has the finite basis property if every closed subset of A is the closure of a finite set. Next we state some alternative definitions of this property.

Theorem 1. *The following conditions on a quasi-ordered set A are equivalent.*

- (1) *Every closed subset of A is the closure of a finite subset;*
- (2) *If B is any subset of A , there is a finite B_0 such that $B_0 \subset B \subset \overline{B_0}$;*
- (3) *Every infinite sequence of elements $\{a_i\}_{i \geq 0}$ of A has an infinite ascending subsequence $a_{i_1} < a_{i_2} < \dots < a_{i_k} < \dots$.*

The next proposition is immediate using condition (3).

Proposition 1. *If $(A_1, \leq_1), \dots, (A_k, \leq_k)$ have the finite basis property than $(A_1 \times \dots \times A_k, \leq)$ has the finite basis property where $(a_1, \dots, a_k) \leq (b_1, \dots, b_k)$ if and only if $a_i \leq_i b_i$ for all $i \in \{1, \dots, k\}$.*

3. THE GRADED IDENTITIES OF B_m

A basis for the graded identities of B_m was determined in [11] using the theory developed in [3]. In this section we recall the necessary results and definitions from these articles.

The polynomials

$$(1) \quad (x_1 z_1) z_2 - x_1 (z_1 z_2),$$

where $z_1 \in \{x_1, y_1\}$ and $z_2 \in \{x_2, y_2\}$, are graded identities for B_m . Moreover we have the following:

Proposition 2. [11, Corollary 21] *The graded identities (1) together with the identity*

$$\sum_{\sigma \in S_{m+1}} (-1)^{\sigma} y_{\sigma(1)} (y_{m+1} y_{\sigma(2)}) \cdots (y_{2m+1} y_{\sigma(m+1)}),$$

form a basis for the ideal of graded identities of the Jordan superalgebra B_m .

A multihomogeneous polynomial $f(x_1, \dots, x_k, y_1, \dots, y_n)$ can be written, modulo $\text{Id}_2(B_m)$, as

$$f(x_1, \dots, x_k, y_1, \dots, y_n) = x_1^{\alpha_1} \dots x_k^{\alpha_k} g(y_1, \dots, y_n)$$

where g is a polynomial on the variables y 's only.

Now we present some results from [3]. A *double tableau* is an array

$$T = \left(\begin{array}{cccc|cccc} p_{11} & p_{12} & \cdots & p_{1m_1} & q_{11} & q_{12} & \cdots & q_{1m_1} \\ p_{21} & p_{22} & \cdots & p_{2m_2} & q_{21} & q_{22} & \cdots & q_{2m_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{k1} & p_{k2} & \cdots & p_{km_k} & q_{k1} & q_{k2} & \cdots & q_{km_k} \end{array} \right),$$

where $m_1 \geq m_2 \geq \dots \geq m_k$ and the p_{ij} and q_{ij} are positive integers. An array obtained from a double tableau by replacing p_{11} by 0 is called a *0-tableau*. If T is a double tableau or a 0-tableau we may form from T the single tableau

$$T' = \left(\begin{array}{cccc} p_{11} & p_{12} & \cdots & p_{1m_1} \\ q_{11} & q_{12} & \cdots & q_{1m_1} \\ p_{21} & p_{22} & \cdots & p_{2m_2} \\ q_{21} & q_{22} & \cdots & q_{2m_2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{k1} & p_{k2} & \cdots & p_{km_k} \\ q_{k1} & q_{k2} & \cdots & q_{km_k} \end{array} \right).$$

We say that T is *doubly standard* if the tableau T' is standard. Recall that the single tableau

$$A = \left(\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1m_1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{km_k} \end{array} \right)$$

is standard if we have the inequalities

- (i) $a_{ij} < a_{il}$, if $l > j$
- (ii) $a_{ij} \leq a_{lj}$, if $l \geq i$.

We associate (see [11]) to a tableau T a polynomial, still denoted by T , of $J(B_m)$

$$T = \prod_{l=1}^k \det |(y_{p_{li}} \cdot y_{q_{lj}})|, \quad i, j = 1, \dots, m_l.$$

If T is a 0-tableau the tableau T_1 obtained from T by deleting the first line is a double tableau and we associate to T the polynomial

$$T = \left(\sum_{\sigma \in S_{m_1}} (-1)^\sigma y_{q_{1\sigma(1)}} (y_{p_{12}} \cdot y_{q_{1\sigma(2)}}) \cdots (y_{p_{1m_1}} \cdot y_{q_{1\sigma(m_1)}}) \right) \cdot T_1.$$

Notation 1. We denote by T_n and T_n^0 the polynomial associated to the tableau $(12 \dots n | 12 \dots n)$ and the 0-tableau $(02 \dots n | 12 \dots n)$ respectively.

Let A be a standard single tableau of the form

$$A = \begin{pmatrix} 1 & 2 & 3 & \cdots & k_1 & \tau_1 & \cdots \\ 1 & 2 & 3 & \cdots & k_2 & \tau_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & & \\ 1 & 2 & 3 & \cdots & k_s & \tau_s & \cdots \\ \tau_{s+1} & & & & & & \end{pmatrix},$$

where $\tau_i > k_i + 1$, $i = 1, \dots, s$ and $\tau_{s+1} > 1$. Since A is standard we have

$$(2) \quad k_1 \geq k_2 \geq \cdots \geq k_s.$$

Let $h_i(A)$ denote the number of times the number $i - 1$ appear in the sequence (2) for $i = 2, \dots, n$. Thus, for example, $h_2(A)$ is the number of rows starting with 1τ , $\tau > 2$, $h_3(A)$ is the number of rows starting with 12τ , $\tau > 3$, etc. Consider the tableau obtained from A by deleting the 1 in each of the $h_2(A)$ rows of A starting with 1τ , $\tau > 2$ and replacing it with 2, then deleting the 1 in each of the $h_3(A)$ rows starting with 12τ , $\tau > 3$, replacing it with a 3, and so on. If we reorder the elements in each row the resulting tableau, which is denoted by $F(A)$ is standard.

Notation 2. Let T be a double tableau and T' the corresponding single tableau associated to T . We denote by $F(T)$ the double tableau for which the corresponding single tableau is $F(T')$.

Let $\lambda_2, \dots, \lambda_n$ be arbitrary scalars. We substitute in the linear combination

$$p = \sum c_i T_i$$

of distinct doubly standard tableau the variable y_1 by $y_1 + \sum_{i=2}^n \lambda_i y_i$. The resulting polynomial in $J(B_m)$ is

$$(3) \quad \sum \lambda_2^{h_2} \lambda_3^{h_3} \cdots \lambda_n^{h_n} P_{h_2 \dots h_n}.$$

We consider the lexicographical order on the tuples (h_2, \dots, h_n) . The next remark is a simple adaptation of the discussion in [3, Section 1] (see also the proof of Lemma 5.4 in [3]).

Remark 1. The polynomial in (3) corresponding to the maximal (h_2, \dots, h_n) is

$$\bar{p} = \sum_j \epsilon_j c_j F(T_j),$$

where $\epsilon_j = \pm 1$, and the sum is taken from the doubly standard tableaux T_j for which $(h_2(T'_j), h_3(T'_j), \dots, h_n(T'_j))$ is maximal. Moreover the doubly standard tableaux $F(T_j)$ are pairwise distinct.

Proposition 3 ([11], Proposition 19). Let M be the subalgebra of $J(B_m)$ generated by the variables in Y . The doubly standard tableau form a basis of the vector space $M_0 = M \cap J(B_m)_0$. Moreover the doubly standard 0-tableau form a basis of the vector space $M_1 = M \cap J(B_m)_1$.

Now we are able to prove our first result. The next remarks will be useful.

Remark 2. Let h, h' be polynomials in M_1 not depending on y_1 . Since the bilinear form in the definition of B_m is nondegenerate the equality $hy_1 = h'y_1$ implies that $h = h'$.

Remark 3. If $h(y_1, \dots, y_{m+1}) \in M_0$ is a polynomial of degree 1 in y_j then there exists $h' \in M_1$ such that $h = h'y_j$. To prove this claim note that by renaming the variables we may assume that $j = 1$. We write h as a linear combination of doubly standard tableaux. The standard nature of the tableaux imply that the leftmost entry in the first row is 1 and this proves the claim.

Proposition 4. Let $g(y_1, \dots, y_m)$ be a multihomogeneous polynomial of $J(B_m)$ that for any scalars $\lambda_2, \dots, \lambda_m$ is invariant under the substitution $y_1 \equiv y_1 + \sum_{i=2}^m \lambda_i y_i$. Then there exists a polynomial $h(y_2, \dots, y_m)$ such that one of the following equalities hold:

- (i) $g(y_1, \dots, y_m) = (T_m)^k h(y_2, \dots, y_m)$,
- (ii) $g(y_1, \dots, y_m) = (T_m)^{k'} T_m^0 h(y_2, \dots, y_m)$,

where $k, k' \geq 0$. Moreover if g lies in $J(B_m)_0$ then the equality that holds is (i).

Proof. We first consider the case $g \in J(B_m)_0$. It follows from the previous proposition that we may write

$$(4) \quad g = \sum c_i T_i, \quad c_i \neq 0$$

as a linear combination of pairwise distinct doubly standard tableau. The invariance of g and Remark 1 imply that $h_2(T'_i) = h_3(T'_i) = \dots = h_m(T'_i) = 0$ for every T_i in (4). Since the variables appearing in g are y_1, \dots, y_m this implies that the entry 1 only appears in the rows of T_i that are equal to $(12 \dots m | 12 \dots m)$. If k_i is the number of such rows we have $2k_i = \deg_{y_1} g$. Let k be the common value of the k_i . For every T_i appearing in (4) we have $T_i = (T_m)^k \hat{T}_i$, where \hat{T}_i is the tableau obtained from T_i by deleting the rows in which the entry 1 appears. Hence g is decomposed as in (i).

Assume now that $g \in J(B_m)_1$ and write gy_{m+1} as in (4). Note that $h_2(T'_i) = h_3(T'_i) = \dots = h_m(T'_i) = 0$ for every T_i appearing gy_{m+1} . Hence the entry 1 only appears in the rows of T_i equal to $(12 \dots m | 12 \dots m)$ or $(12 \dots m | 2 \dots m + 1)$. This last row appears at most once, it appears if $\deg_{y_1} g$ is odd and does not appear otherwise. Let k_i be the number of rows in T_i that are equal to $(12 \dots m | 12 \dots m)$. If $(12 \dots m | 23 \dots m + 1)$ does not appear in T_i we have $2k_i = \deg_{y_1} g$. Therefore in this case for every T_i appearing in gy_{m+1} we have $T_i = (T_m)^k \hat{T}_i$, where $k = \deg_{y_1} g / 2$. Hence $gy_{m+1} = (T_m)^k h(y_2, \dots, y_{m+1})$ and it follows from Remark 2 and Remark 3 that g is as in (i). If $(12 \dots m | 23 \dots m + 1)$ appears in some T_i we have $2k_i + 1 = \deg_{y_1} g$ and thus it appears in every T_i . The polynomial associated to the tableau $(12 \dots m | 23 \dots m + 1)$ is $\pm T_m^0 y_{m+1}$. In this case $gy_{m+1} = ((T_m)^k T_m^0 h(y_2, \dots, y_m)) y_{m+1}$, where $k = (\deg_{y_1} g - 1) / 2$. From this last equality and Remark 2 we conclude that g is as in (ii). \square

4. THE MAIN RESULTS

We denote by M_m the subalgebra of $J(B_m)$ generated by y_1, \dots, y_m . Let W denote the subspace of $J(B_m)$ generated by y_1, \dots, y_m and GL_m the group of invertible linear transformations of W . The canonical action of GL_m on M_m turns this subalgebra into a GL_m -module. We refer the reader to [5] for basic facts regarding polynomial representations of GL_m and applications to algebras with polynomial identities.

Lemma 1. [5, Theorem 2.2.11] *Let $g(y_1, \dots, y_m)$ be a multihomogeneous polynomial of M_m . The GL_m -module generated by $g(y_1, \dots, y_m)$ is an irreducible module if and only if for any scalars λ_{ij} $1 \leq i < j \leq m$ it is invariant under the substitutions*

$$(5) \quad y_j \equiv \lambda_{1j}y_1 + \dots + \lambda_{j-1j}y_{j-1} + y_j \quad j = 1, 2, \dots, m.$$

Next we apply this lemma and Proposition 4 to determine generators of the irreducible submodules of M_m . The following notation will be useful.

Notation 3. *Given a polynomial $g(y_1, \dots, y_m)$ in M_m we denote $\widehat{g}(y_1, \dots, y_m)$ the polynomial $g(y_m, \dots, y_1)$. Denote by S_m the polynomial corresponding to the tableaux $(012 \dots m-1|12 \dots m)$. We have $\widehat{T_m^0} = \pm S_m$.*

Note that the polynomial $g(y_1, \dots, y_m)$ in M_m is invariant under the substitutions

$$y_m \equiv \lambda_1 y_1 + \dots + \lambda_{m-1} y_{m-1} + y_m,$$

for any $\lambda_i \in K$ if and only if $\widehat{g}(y_1, \dots, y_m) = g(y_m, \dots, y_1)$ is invariant under the substitutions

$$(6) \quad y_1 \equiv y_1 + \sum_{i=2}^m \lambda_i y_i,$$

for any $\lambda_i \in K$.

Corollary 1. *The polynomial $g(y_1, \dots, y_m) \in M_m$ generates an irreducible GL_m -submodule of M_m if and only if it is up to multiplication by a scalar one of the polynomials*

$$(7) \quad (S_m)^{\delta_m} \dots (S_1)^{\delta_1} (T_m)^{k_m} \dots (T_1)^{k_1}$$

where $k_1, \dots, k_m \geq 0$, $\delta_l \in \{0, 1\}$ and $\delta_l \neq 0$ for at most one index $l \in \{1, \dots, m\}$.

Proof. We prove the result by induction on m . If $m = 1$ the result is obvious. Now let $g(y_1, \dots, y_m) \in M_m$ generate an irreducible submodule. It follows from Lemma 1 that \widehat{g} is invariant under the substitutions (6). It follows from Proposition 4 that we have two possibilities: (i) $\widehat{g} = (T_m)^k h(y_2, \dots, y_m)$ or (ii) $\widehat{g} = (T_m)^{k'} T_m^0 h(y_2, \dots, y_m)$. In the first case we obtain $g = (T_m)^k h'(y_1, \dots, y_{m-1})$. It is clear that h' is invariant under the substitutions (5) and the result follows. In the second case $g = \pm (T_m)^{k'} S_m h'(y_1, \dots, y_{m-1})$ with h' invariant under the substitutions (5). Note that this occurs only if g lies in $J(B_m)_1$ and in this case h' lies in $J(B_m)_0$. The induction hypothesis applied to h' implies that it is up to scalar of the form $(T_{m-1})^{k_{m-1}} \dots (T_1)^{k_1}$. \square

Example 1. *The algebra B_2 of 2×2 symmetric matrices over K with the Jordan product $a \circ b = (ab + ba)/2$ is a Jordan algebra of a nondegenerate symmetric bilinear form on a vector space of dimension 2. In this case $g(y_1, y_2) \in M_2$ generates an irreducible GL_2 -submodule of M_2 if and only if it is up to multiplication by a scalar one of the polynomials*

$$(y_1^2 y_2^2 - (y_1 y_2)^2)^{k_2} y_1^{2k_1} \text{ if } g \in J(B_2)_0;$$

$$(y_1^2 y_2^2 - (y_1 y_2)^2)^{k_2} y_1^{2k_1+1} \text{ or } (y_1^2 y_2^2 - (y_1 y_2)^2)^{k_2} (\bar{y}_1 y_1 \bar{y}_2) y_1^{2k_1} \text{ if } g \in J(B_2)_1.$$

Recall that any result on homogeneous polynomial identities obtained in the language of representations of the general linear group is equivalent to a corresponding result on multilinear polynomial identities obtained in the language of representation of the symmetric group. The complete linearization of a highest weight vector associated to an irreducible $GL_1 \times GL_m$ -module generates an irreducible $S_k \times S_n$ -module. As a consequence we have the description of the 2-graded cocharacter sequence of B_m .

Theorem 2. *Let*

$$\chi_{k,n}(B_m) = \sum_{\lambda \vdash k, \mu \vdash n} m_{\lambda,\mu} \chi_\lambda \otimes \chi_\mu$$

be the (k, n) -th cocharacter of B_m . Then, for every $\lambda \vdash k$ and $\mu \vdash n$, $m_{\lambda,\mu} \leq 1$. Moreover, $m_{\lambda,\mu} = 1$ if and only if $\lambda = (k)$, $\mu = (r_m, \dots, r_1)$ is a partition of n such that at most one r_i is odd.

Now we proceed to prove that B_m has the Specht property. We will need the following:

Lemma 2. *Let $\widetilde{T}_k, \widetilde{S}_k$, $k = 1, 2, \dots$ be polynomials in $J(Z)$ such that the image of $\widetilde{T}_k, \widetilde{S}_k$ under the canonical homomorphism are the polynomials T_k, S_k in $J(B_m)$ respectively. The set of polynomials*

$$(8) \quad x_1^{k_0} (\widetilde{S}_m)^{\delta_m} \dots (\widetilde{S}_1)^{\delta_1} (\widetilde{T}_m)^{k_m} \dots (\widetilde{T}_1)^{k_1},$$

where $k_0, \dots, k_m \geq 0$, $\delta_l \in \{0, 1\}$ and $\delta_l \neq 0$ for at most one index $l \in \{1, \dots, m\}$, has the following property: given any subset S there exists a finite subset $\widehat{S} \subseteq S$ such that any polynomial in S is the consequence of a polynomial in $\widehat{S} \cup Id_2(B_m)$.

Proof. To prove this we define in \mathbb{N}^{2m+1} the partial order \leq as follows:

$$(\delta_1, \dots, \delta_m, k_0, \dots, k_m) \leq (\delta'_1, \dots, \delta'_m, k'_0, \dots, k'_m) \text{ if } \delta_i \leq \delta'_i, k_j \leq k'_j \text{ for all } i, j.$$

Clearly the above inequality implies that $x_1^{k'_0} (S_m^0)^{\delta'_m} \dots (S_1^0)^{\delta'_1} (T_m)^{k'_m} \dots (T_1)^{k'_1}$ is obtained from $x_1^{k_0} (S_m)^{\delta_m} \dots (S_1)^{\delta_1} (T_m)^{k_m} \dots (T_1)^{k_1}$ by multiplication by a suitable element of $J(B_m)$. By Proposition 1 the set \mathbb{N}^{2m+1} with the partial order above has the finite basis property and this implies the result. \square

Next we prove our main result.

Theorem 3. *The ideal $Id_2(B_m)$ of \mathbb{Z}_2 -graded identities of $B_m = K \oplus V$ has the Specht property.*

Proof. Let I be a T_2 -ideal containing $Id_2(B_m)$. Henceforth we say that two sets of polynomials S and S' in $J(Z)$ are equivalent if $S \cup Id_2(B_m)$ and $S' \cup Id_2(B_m)$ generate the same T_2 -ideal. We claim that for every multilinear polynomial g in I there exists a set of polynomials $S_g \subseteq I$ of polynomials of the form (8) such that $\{g\}$ and S_g are equivalent. Let $S = \cup S_g$, where the union is over all multilinear polynomials g in I . Since the field is of characteristic zero this implies that I is generated by $S \cup Id_2(B_m)$. Hence the previous lemma and Proposition 2 imply the theorem. Let $g(x_1, \dots, x_k, y_1, \dots, y_n)$ be a multilinear element in I . We work in the relatively free algebra $J(B_m)$ and denote by $\varphi : J(Z) \rightarrow J(B_m)$ the canonical homomorphism. The group S_n acts on the odd variables of $\varphi(g)$. We decompose this module, denoted by M , into a direct sum of irreducible modules

$$M = M_1 \oplus \dots \oplus M_q,$$

and let p_i be a generator of the S_n -module M_i . A set of q polynomials $\{P_1, \dots, P_q\}$ such that $\varphi(P_i) = p_i$ is equivalent to $\{g\}$. The result is proved once we prove that each P_i is equivalent to a finite set of polynomials of the form (8). Let $\lambda \vdash n$ be the partition of n corresponding to M_i . We write $p_i = x_1 \cdots x_k p'_i(y_1, \dots, y_n)$ and note that the S_n -module generated by p'_i is isomorphic as an S_n -module to M_i . Since $\dim V = m$ we conclude that $\lambda_{m+1} = 0$. Then p'_i is symmetric in m disjoint sets of variables and we may identify the variables in each set to obtain a multihomogeneous polynomial $q_i(y_1, \dots, y_m)$. The complete linearization of q_i is p'_i . We decompose the GL_m -module generated by q_i and let q_i^1, \dots, q_i^r be generators of its irreducible components. Thus Corollary 1 implies that each q_i^j is up to scalar a polynomial of the form (7) and hence $x_1^k q_i^j$ is the image under the canonical homomorphism of a polynomial Q_i^j of the form (8). The linearization polynomial $x_1^k q_i$ is up to multiplication by scalar p_i . The set $S_i = \{Q_i^1 \dots Q_i^r\}$ is equivalent to $\{P_i\}$. Hence $S_g = S_1 \cup \dots \cup S_q$ is equivalent to $\{g\}$. \square

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